Propagation of an envelope soliton in a medium with spatially varying dispersion

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We study analytically and numerically the motion of a nonlinear Schrodinger soliton in a medium with spatially modulated dispersion. The cases of periodic and random modulations of the dispersive term are considered. In the former, numerical simulations for small velocities show a good agreement with the adiabatic equations. When the velocity is increased the soliton emits linear waves and we calculate their spectral density and show the existence of a resonant condition connecting the amplitude and velocity of the soliton to the wavelength of the modulation. The important application of steering a spatial soliton in an array of tunnel coupled planar waveguides with variable coupling is considered. [S1063-651X(97)08104-X]

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I. INTRODUCTION

Recently some authors have investigated the dynamics of an envelope soliton in a medium with a time-dependent dispersion [1–3]. This connects with the important application of soliton propagation in optical fibers and in optical loop devices. The dynamics of a single soliton in a medium with a periodic and random variation of dispersion (or nonlinearity) has been studied [1,2] and the resonant emission of soliton and radiative damping has been estimated. This is important for understanding the motion of a soliton in a fiber optical loop with variable dispersion. For chirped pulses, the case of a dispersion periodically modulated in time was considered in [3] using a variational approach. In some cases the width grows to infinity, indicating a destruction of the pulse. Recently the influence of a strong temporal dispersive δ-like perturbation on a soliton propagating in an optical fiber has been studied in [4]. In the case of the propagation of spatial optical solitons it is important to study the influence of a spatial variation of the dispersion. Such types of problems appear, for example, when studying the propagation of a soliton beam in an array of planar optical waveguides with variable coupling [5,7]. This system has attracted attention recently because of its importance for all optical processing. It is important to note that in the linear limit, it can exhibit localization as in the Anderson or Lifschitz models if the dependence that is random is the refraction index or the separation between waveguides [5]. An array of nonlinear waveguides seems to be a unique system for the observation of the competition between nonlinearity and disorder and the experimental observation of the influence of the nonlinearity on Anderson localization. Usually the static disorder induced by the linear random potential is studied. An equally important interest for soliton perturbation theory is to investigate the influence of random dispersive perturbations. Taking into account these facts, in this work we perform an analytical and numerical investigation of the influence of a spatial modulation of the dispersion on the process of envelope soliton propagation. The cases of periodic and random modulations of dispersion are considered. In Sec. II the adiabatic dynamics of solitons in the medium with a periodic variation of dispersion is considered, phase portraits, fixed point analysis and comparison with the full numerics for the partial differential equation (PDE) are performed. Section III describes the adiabatic dynamics of a soliton in a medium with a randomly fluctuating dispersion for short times, and presents the behavior of the mean values of the soliton parameters. In Sec. IV the radiative processes of solitons connected with propagation for large velocity are studied for the periodic modulation case, the spectral density of emission is found together with a resonant condition. In Sec. V we consider the physical application of this model to the propagation of spatial optical solitons in arrays of planar nonlinear optical waveguides with variable coupling.

II. ADIABATIC DYNAMICS OF NLS SOLITON IN THE MEDIUM WITH THE PERIODICALLY MODULATED DISPERSION

We will study in this work the nonlinear Schrödinger (NLS) soliton dynamics in media with spatially changing dispersion. The problem is described by a modified NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = V(x) u_{xx}, \tag{1}$$

in which the dispersion term is modulated in space. In real physical situations we must add other terms such as $V(x)u$ and $V_x u_x$ (see Sec. V). The linear potential case $V(x)u$ has been considered by Scharf and Bishop [6]. Here we will study the simple model given by Eq. (1).

Let us consider the propagation in such a medium of the single soliton solution

$$u_s(x,t) = 2i\eta \,\mathrm{sech}(z) \exp(-i\psi), \tag{2}$$

$$\psi = 2\xi x + \phi, \quad \phi = 4(\xi^2 - \eta^2)t, \tag{3}$$

$$z = 2\eta(x - \zeta), \quad \zeta = -4\xi t. \tag{4}$$

Assuming $V(x)=\epsilon\sin(ax)$, where $L_a=2\pi/a$ is the period of dispersion modulation, and applying the equations of the adiabatic approximation [8,9], we obtain the following system of equations for the amplitude η, velocity ξ, the center of the soliton ζ, and phase ϕ, correspondingly:

$$\frac{d\eta}{dt}=\frac{\pi\epsilon a^2\xi\cos(a\zeta)}{\sinh(\pi a/4\eta)}, \tag{5}$$

$$\frac{d\xi}{dt}=\frac{\pi a^2\epsilon\eta}{2\sinh(\pi a/4\eta)}\left[\frac{1}{3}\left(1-\frac{a^2}{8\eta^2}\right)-\frac{\xi^2}{\eta^2}\right]\cos(a\zeta), \tag{6}$$

$$\frac{d\zeta}{dt}=-4\xi\left\{1-\frac{\pi a\epsilon\sin(a\zeta)}{2\eta\sinh(\pi a/4\eta)}\times\left[1-\frac{\pi a}{8\eta}\coth\left(\frac{\pi a}{4\eta}\right)\right]\right\}, \tag{7}$$

$$\frac{d\phi}{dt}=4(\xi^2-\eta^2)+\frac{\pi a\epsilon\sin(a\zeta)}{2\eta\sinh(\pi a/4\eta)}\left[\frac{a^2}{12\eta^2}+\left(1-\frac{\xi^2}{\eta^2}-\frac{\pi a}{6\eta}-\frac{\pi a^3}{96\eta^3}\right)\coth\left(\frac{\pi a}{4\eta}\right)\right]+\frac{\pi a^2\epsilon\zeta\cos(a\zeta)}{12\eta\sinh(\pi a/4\eta)}\left[3\frac{\xi^2}{\eta^2}+\frac{a^2}{8\eta^2}-1\right]. \tag{8}$$

These equations are complicated to analyze so we first consider the case of small velocities $\xi_0\sim\epsilon$. In that approximation one can then neglect the variation of amplitude with time and let $\eta=\eta_0$. Then from Eqs. (5)–(8) one can separate the equations for the velocity ξ and the coordinate of the soliton center ζ:

$$\frac{d\xi}{dt}=\frac{\pi a^2\epsilon\eta}{6\sinh(\pi a/4\eta)}\left(1-\frac{a^2}{8\eta^2}\right)\cos(a\zeta), \tag{9}$$

$$\frac{d\zeta}{dt}=-4\xi. \tag{10}$$

Changing variables by $\zeta\to\zeta+\pi/2a, t\to t\sqrt{a}$ we have from Eqs. (9) and (10)

$$\frac{d^2\zeta}{dt^2}+\omega_0^2\sin(a\zeta)=0, \tag{11}$$

which is the equation of a pendulum with a frequency given by

$$\omega_0^2=\frac{2\pi a\epsilon\eta}{3\sinh(\pi a/4\eta)}\left(1-\frac{a^2}{8\eta^2}\right). \tag{12}$$

This shows that the soliton is oscillating slowly in the effective potential well with the frequency $\omega_0\sim\exp(-\pi a/4\eta)$ [10,6]. The effective potential is exponentially small for the broad solitons for which $\pi a/4\eta\gg1$.

Let us find the fixed points from the system (5)–(8). We will not consider Eq. (8) which is uncoupled from the system. It can be seen that the condition for $d\zeta/dt=0$ is $\xi=0$ because the term in the brackets is always strictly positive when $\epsilon<1$. Thus the fixed points of the system (5)–(7) are such that

$$\xi=0 \quad \text{and} \quad a^2=8\eta^2 \quad \text{or} \quad \zeta=\frac{\pi}{2a}+k\frac{\pi}{a}, \tag{13}$$

where k is an integer. For these two sets of fixed points the Jacobian matrix is singular so that nothing can be said about the stability by considering the linearized system. Numerical results indicate, however, that both sets of fixed points are unstable.

It is important to notice that the perturbation on the right-hand side (rhs) of Eq. (1) does not allow the conservation of the number of particles $N=\int_{-\infty}^{\infty}|u|^2dx$, momentum $P=(i/2)\int_{-\infty}^{\infty}(u_xu^*-u_x^*u)dx$, or energy $E=\int_{-\infty}^{\infty}(|u|^4-|u_x|^2)dx$. Following the method suggested by Karpman [11], one can find the evolution of these quantities under the action of the perturbation.

$$\frac{dN}{dt}=2\epsilon\int_{-\infty}^{\infty}V_x\mathrm{Im}(u_x^*u)dx, \tag{14}$$

$$\frac{dP}{dt}=\epsilon\int_{-\infty}^{\infty}V_x|u_x|^2dx, \tag{15}$$

$$\frac{dE}{dt}=4\epsilon\int_{-\infty}^{\infty}V|u|^2\mathrm{Im}(u_{xx}u^*)dx, \tag{16}$$

We have compared the evolution of the soliton parameters in the adiabatic approximation given by Eqs. (5)–(8) with the full numerical solution of the PDE (1), which was solved using a method of lines in which the space derivative is discretized using finite differences with Dirichlet boundary conditions, and the solution advanced in time by an ordinary differential equation solver: we have used the variable step Runge-Kutta method of order 5 from [12]. Results were checked by monitoring the evolution of the number of particles, momentum, and energy as defined above and comparing their numerically computed derivatives with the right-hand sides in the formulas (14)–(16), in all cases the agreement was better than 0.005 in relative difference. We also checked the results by doubling the number of grid points. The soliton parameters η and ζ were then estimated by least-squares fitting the modulus of the numerically computed solution by the expression given in Eq. (2). ξ was then estimated from Eq. (4).

Figure 1 shows the time evolution of η and ζ for the PDE as a solid line and for the adiabatic equations as a dashed line for $a=1$ and a soliton with $2\eta=1$ with initial position $\zeta_0=10\pi$, for which $\sin(a\zeta_0)=0$ and zero initial velocity. This initial condition is such that the perturbation is zero initially. This is important because if the perturbation is not zero for $t=0$, the initial condition should be modified to take into account the new balance between dispersion and nonlinearity. Figure 1 shows an excellent agreement for ζ, showing that the soliton is oscillating in the potential well with a period of about 41.6, which is slighly larger than the period derived from Eq. (12) because of the variation of η. There is a 2% relative mismatch for η between the PDE and

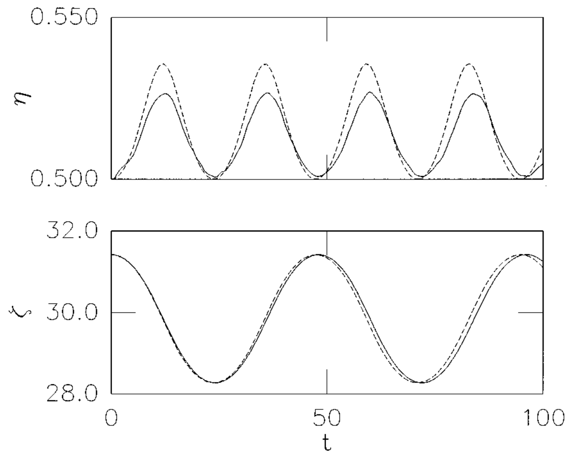

FIG. 1. Time evolution of η (top) and ζ (bottom) for the PDE (solid line) and the adiabatic equations (dashed line). The values of the parameters are $a=1$, $\epsilon=0.1$ and the initial conditions are $\eta_0=0.5$, $\zeta_0=10\pi$, and $\xi_0=0$.

the adiabatic formulas. Indeed we have found that the η estimated from the PDE is consistently smaller than the one predicted by the adiabatic equations when the modulation period is large compared to the soliton width.

Figure 2 shows the phase portraits (η,ζ) and (ζ,ξ) for the PDE and adiabatic equations corresponding to the evolution described in Fig. 1. The analogy with the pendulum phase-space is clear from the (ζ,ξ) phase-portrait where the central fixed point has been represented.

III. MEDIA WITH RANDOM SPATIAL DISPERSION

We will consider in this section the propagation of spatial solitons in a medium with random spatial dispersion in the direction of propagation. For that we have modeled V with a random function uniformly distributed between -1 and $+1$ [13] multiplied by a given amplitude factor. Figure 3 shows a soliton moving in one realization of the random potential for different times. The amplitude of the potential is 0.1 and the parameters of the soliton initially are $\eta_0=0.5$, $\zeta_0=10\pi$, and $\xi_0=0.05$. It can be seen that the pulse first moves toward $x=0$ and then back toward the large values of x, this is due to the soliton encountering a potential barrier it cannot overcome. At the end of the paper we will calculate the average pinning potential for a soliton in an array of randomly coupled waveguides.

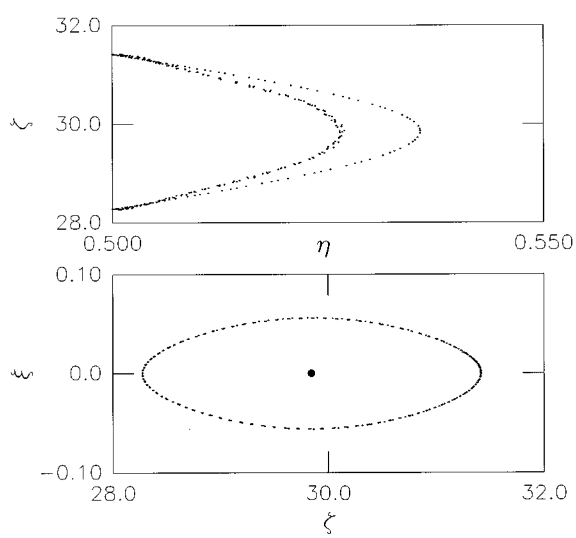

FIG. 2. Phase portraits (η,ζ) (top) and (ζ,ξ) (bottom) for the PDE and adiabatic evolutions corresponding to Fig. 1.

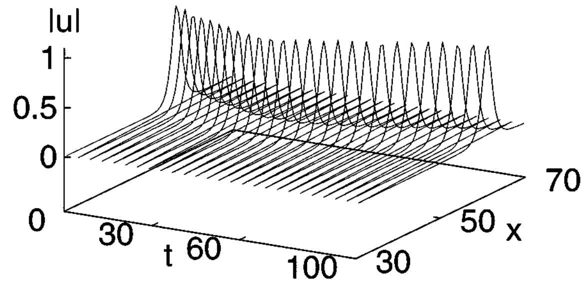

FIG. 3. Profile of $|u|$ (x,t) for a soliton propagating in a medium with random dispersion. The parameters are $\epsilon=0.1$, $\eta_0=0.5$, $\xi_0=0.05$, $\zeta_0=10\pi$.

For a large number of realizations one can examine the average of the soliton parameters. For this we chose the parameters $\eta_0=0.5$, $\epsilon=0.1$ and small velocities for the soliton $\xi_0=0.05$ and $\xi_0=0.15$. A common feature of these two sets of experiments is that the average value of η varies by less than 1% over the time interval considered ($t<30$). The time variations of $\langle\zeta\rangle$ and $\langle\xi\rangle$ are presented in Fig. 4 from top to bottom. 400 realizations were necessary for the average value of the velocity $\langle\xi\rangle$ to reach a stationary value within a few percent. The picture shows that $\langle\zeta\rangle$ follows an almost linear behavior given to a good approximation by $\langle\zeta\rangle=-4\langle\xi\rangle t+\zeta_0$ while $\langle\xi\rangle$ decreases by about 10% and exhibits oscillations of period about 10. Let us show that some of these results can be explained with the help of the adiabatic equations for the soliton parameters

$$\frac{d\eta}{dt}=8\epsilon\xi\eta^2\int_{-\infty}^{+\infty}\cosh^{-2}z\tanh z V\left(\frac{z}{2\eta}+\zeta\right)dz, \qquad (17)$$

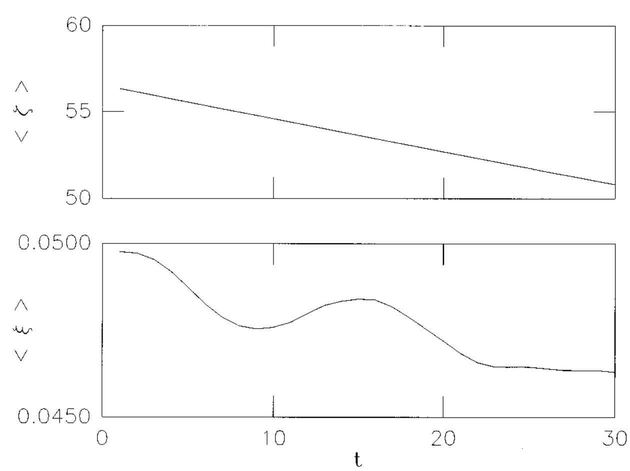

FIG. 4. Time evolution of $\langle\zeta\rangle$ (top) and $\langle\xi\rangle$ (bottom) for the PDE solution. The average was done on 400 realizations. The values of the parameters are $\epsilon=0.1$ and the initial conditions are $\eta_0=0.5$, $\zeta_0=10\pi$, and $\xi_0=0.05$.

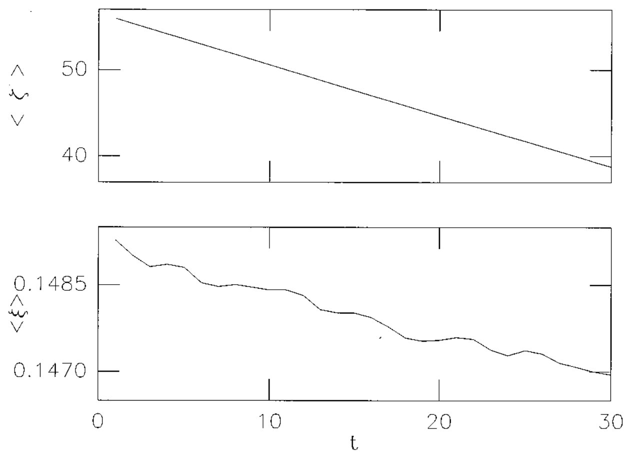

FIG. 5. Same as Fig. 4 except that the initial velocity is $\xi_0=0.15$.

$$\frac{d\xi}{dt}=4\epsilon\eta^3\int_{-\infty}^{+\infty}\cosh^{-2}z(-1+2\tanh^2 z)\tanh z\, V\left(\frac{z}{2\eta}+\zeta\right)dz -4\epsilon\xi^2\eta\int_{-\infty}^{+\infty}\cosh^{-2}z\tanh z\, V\left(\frac{z}{2\eta}+\zeta\right)dz, \tag{18}$$

$$\frac{d\zeta}{dt}=-4\xi+4\epsilon\xi\int_{-\infty}^{+\infty}z\cosh^{-2}z\tanh z\, V\left(\frac{z}{2\eta}+\zeta\right)dz. \tag{19}$$

The numerical results show that η can be assumed to be constant so that using the change of variables $z\rightarrow z/2\eta+\zeta$ the system reduces to the two equations

$$\frac{d\xi}{dt}=4\epsilon\eta^3\int_{-\infty}^{+\infty}V(x)f_s[2\eta(x-\zeta)]dx -4\epsilon\xi^2\eta\int_{-\infty}^{+\infty}V(x)g_s[2\eta(x-\zeta)]dx, \tag{20}$$

$$\frac{d\zeta}{dt}=-4\xi+O(\epsilon), \tag{21}$$

where $f_s(z)=\cosh^{-2}z(-1+2\tanh^2 z)\tanh z$ and $g_s(z)=\cosh^{-2}z\tanh z$ The second equation gives by averaging the evolution of $\langle\zeta\rangle$ $\langle\zeta\rangle(t)=-4\langle\xi\rangle t+\zeta_0$, which is the result observed in the numerical simulations. For $t<30$ this is verfied with a 3% relative error.

The evolution of ξ is complicated: the first term on the rhs of Eq. (20) can be averaged out assuming that η is constant but the second term cannot be computed because of the presence of ξ. Therefore we do not have at this time an analytic estimate of the evolution of $\langle\xi\rangle$. The range of this evolution is much smaller if the initial velocity ξ_0 is larger ($\xi_0=0.15$) as shown in Fig. 5 where the decrease of $\langle\xi\rangle$ is about 1%. Note that the behavior of ζ is again given by $\langle\zeta\rangle(t)=-4\langle\xi\rangle t+\zeta_0$.

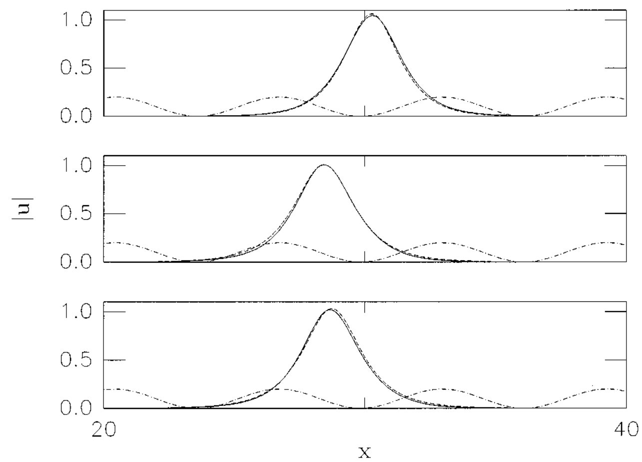

FIG. 6. Profile of $|u|$ as a function of x for different times $t=10$ (top), $t=20$ (middle), $t=30$ (bottom) corresponding to Fig. 1. $V(x)$ has been drawn as dashed lines and shifted vertically for clarity.

To conclude, it seems that the decay of the velocity of the soliton for small velocity is a resonant effect. To study a simpler situation we now consider the emission of linear waves by the soliton in the case of a periodic modulation of the dispersion. We will find a resonance condition connecting the soliton parameters with the wave length of the modulation. This resonance condition explains many numerical results obtained previously by other authors.

IV. THE EMISSION OF WAVES BY SOLITONS IN A MEDIUM WITH A PERIODICALLY MODULATED DISPERSION

Together with the adiabatic dynamics, which is the correction to the discrete spectrum of the associated linear spectral problem due to the perturbation, the correction to the continuum part of spectrum also gives a contribution to the solution. This continuum correction gives two kinds of contributions: the first one corresponds to the correction localized on the soliton and is responsible for the reconstruction of the soliton profile while the second represents an oscillating field that has a part corresponding to the emission of linear waves by the soliton. Below we find both parts using the perturbation theory based on the inverse scattering transform [8,9,15].

Let us calculate initially the correction localized on the soliton δu_{loc}. According to [8] the asymptotic behavior of δu_{loc} is defined by the expression

$$\delta u_{\mathrm{loc}}=2i\eta w_{\mathrm{loc}}\exp(-i\psi),$$

where

$$w_{\mathrm{loc}}^{+}=\frac{\epsilon z^2\exp(-z)}{32i\eta^3}\int_{-\infty}^{\infty}\left[R\exp\left(-z'-i\frac{\xi}{\eta}z'-i\delta\right)+R^{*}\exp\left(z+i\frac{\xi}{\eta}z'+i\delta\right)\right]\mathrm{sech}(z')dz' \tag{22}$$

for $z\rightarrow+\infty$ and

$$w_{\rm loc}^{-}(z)=\frac{\epsilon z^2\exp(z)}{32i\eta^3}\int_{-\infty}^{\infty}\left[R\exp\left(z'-i\frac{\xi}{\eta}z'-i\delta\right)+R^*\exp\left(-z'+i\frac{\xi}{\eta}z'+i\delta\right)\right]{\rm sech}(z')dz' \tag{23}$$

for $z\to-\infty$ and where $R=-iV(x)u_{xx}=-i\sin(ax)u_{xx}$ and $\delta=-\phi-2\xi\zeta+\pi/2$. Substituting into Eqs. (22) and (23) the expression for the perturbation

$$V(x)u_{xx}=\epsilon\sin(ax)8i\eta^3{\rm sechz}\left[1-\frac{\xi^2}{\eta^2}-2{\rm sech}^2(z)+2i\frac{\xi}{\eta}\tanh(z)\right]\exp(-i\psi), \tag{24}$$

we find for the asymptotic behavior of the localized correction for $|z|\gg1$

$$w_{\rm loc}^{+}(z)\approx\frac{\epsilon z^2\pi a^2\cos(a\zeta)\exp(-z)}{16\eta^2\sinh(\pi a/4\eta)}\left\{-1-\frac{\xi^2}{\eta^2}+i\left[-\frac{2\xi}{\eta}+\frac{2}{3}\left(\frac{a^2}{16\eta^2}+1\right)\right]\right\}, \tag{25}$$

$$w_{\rm loc}^{-}(z)\approx\frac{\epsilon z^2\pi a^2\cos(a\zeta)\exp(z)}{16\eta^2\sinh(\pi a/4\eta)}\left\{-1+\frac{\xi^2}{\eta^2}+i\left[-\frac{2\xi}{\eta}+\frac{2}{3}\left(\frac{a^2}{16\eta^2}+1\right)\right]\right\}. \tag{26}$$

The adiabatic evolution of the soliton parameters together with the localized corrections describe well the PDE solution at low velocity as shown in Fig. 6, which displays the solution (PDE in full line, adiabatic plus correction in dashed line) as a function of x for times $t=10,20,30$ corresponding to the parameters of Figs. 1 and 2.

When the initial velocity is increased, the soliton emits radiation. Let us estimate the spectral density of waves emitted by the soliton. The energy of emission is

$$E_{\rm rad}=\frac{2}{\pi}\int_{-\infty}^{\infty}\ln[|a(\lambda)|^{-2}]d\lambda, \tag{27}$$

with $|a|^2+|b|^2=1$. Then for $\epsilon\ll1$, $\ln(|a|^{-2})\approx|b|^2$ and

$$E_{\rm rad}=\frac{2}{\pi}\int_{-\infty}^{\infty}|b(\lambda)|^2d\lambda,\quad t\gg1. \tag{28}$$

The wave number of the emitted waves is $k=2\lambda$ and their frequency $\omega(\lambda)=4\lambda^2$. The spectral density of waves is just the derivative of the integrand of $E_{\rm rad}$ with respect to λ. The Jost coefficient $b(\lambda,t)$ satisfies the equation [11]

$$\frac{\partial b}{\partial t}=-4i\lambda^2b+\frac{i\epsilon\exp(i\delta-2i\lambda\zeta)}{2\eta[\Delta^2+\eta^2]}A(\lambda,\xi,\eta), \tag{29}$$

where

$$A(\lambda,\xi,\eta)=\int_{-\infty}^{\infty}\exp\left(-i\frac{\Delta z}{\eta}\right)[(\Delta-i\eta{\rm tanhz})^2R-\eta^2{\rm sech}^2zR^*]dz, \tag{30}$$

and $\Delta=\lambda+\xi$.

Substituting the expression for R from Eq. (24) into Eq. (30), we get after some tedious calculations

$$A(\lambda,\xi,\eta)=\frac{4\pi\eta e^{-ia\zeta}}{\cosh(\pi\Delta_+/2\eta)}F_+(\lambda,\xi,\eta,\zeta)+\frac{4\pi\eta e^{ia\zeta}}{\cosh(\pi\Delta_-/2\eta)}F_-(\lambda,\xi,\eta,\zeta), \tag{31}$$

where

$$F_\pm=\pm\left(-\Delta^2\xi^2+2\xi\eta^2\Delta+\frac{\eta^4}{2}\right)+\Delta_\pm\left(2\xi\Delta^2-\frac{4}{3}\eta^2\Delta+2\xi^2\Delta-\frac{5}{3}\xi\eta^2\right)\pm\Delta_\pm^2\left(\Delta^2+\frac{5}{2}\eta^2+2\xi\Delta\right)+\Delta_\pm^3\frac{2}{3}\Delta$$

and $\Delta_\pm=a/2\pm\Delta$.

To compute $b(\lambda)$ and $P(\lambda)$ it is useful to introduce $\overline{b}=e^{4i\lambda^2t}b$, which satisfies the ordinary differential equation

$$\frac{\partial\overline{b}}{\partial t}=\frac{i\epsilon A\exp[i(\delta-2\lambda\zeta+4\lambda^2t)]}{2\eta(\Delta^2+\eta^2)}$$

and is such that $\mathrm{Re}(b^*\partial b/\partial t)=\mathrm{Re}(\overline{b^*}\partial\overline{b}/\partial t)$.

Using the time dependence for the parameters ζ, ϕ obtained from the adiabatic equations (5)–(8) one obtains

$$\frac{\partial\overline{b}}{\partial t}=\frac{i\epsilon2\pi}{\Delta^2+\eta^2}\left[\frac{F_+e^{i\phi_+t}}{\cosh(\pi\Delta_+/2\eta)}+\frac{F_-e^{i\phi_-t}}{\cosh(\pi\Delta_-/2\eta)}\right], \tag{32}$$

where $\phi_\pm=4(\xi^2+\eta^2)+4\xi(2\lambda\pm a)+4\lambda^2$.

Because of the oscillatory nature of this function, it needs to be integrated after multiplication by an integrating factor $e^{-\alpha t}$. Taking the limit $\alpha\to0$, we obtain for the spectral power density

$$P(\lambda)=2\mathrm{Re}\left(\overline{b}\frac{\partial\overline{b^*}}{\partial t}\right)=\frac{8\pi^3\epsilon^2}{(\Delta^2+\eta^2)^2}\left[\frac{F_+^2\delta_D(\phi_+)}{\cosh^2(\pi\Delta_+/2\eta)}+\frac{F_-^2\delta_D(\phi_-)}{\cosh^2(\pi\Delta_-/2\eta)}\right], \tag{33}$$

where δ_D is the Dirac delta function.

For positive ξ and a, ϕ_+ is never 0 but ϕ_- is 0 for $\lambda=\lambda_2^1$ where

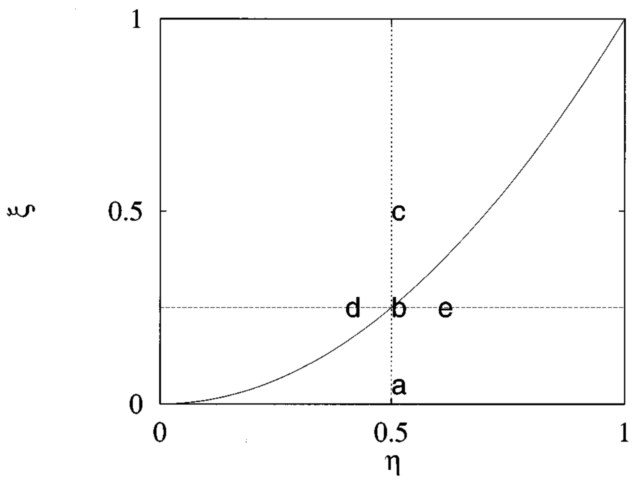

FIG. 7. Plane (η,ξ) showing the resonance condition $a\xi-\eta^2=0$ for $a=1$.

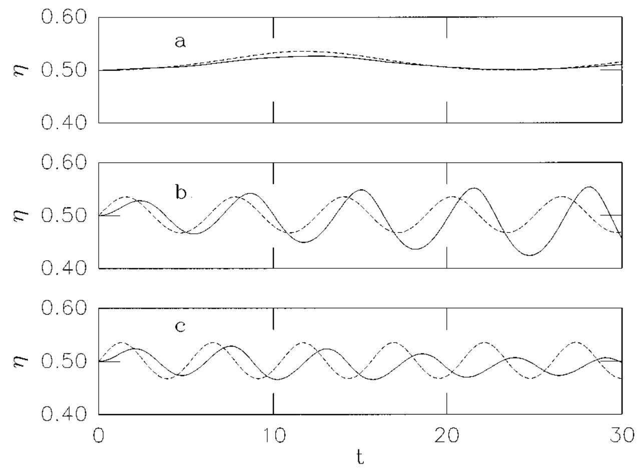

FIG. 8. Time evolution of η for the PDE (solid line) and the adiabatic equations (dashed line) for the positions a, b, and c indicated in Fig. 7 corresponding to initial velocities $\xi_0=0, 0.25$, and 0.3, respectively. The values of the parameters are $a=1$, $\epsilon=0.1$ and the initial conditions are $\eta_0=0.5$, $\zeta_0=10\pi$.

$$\lambda_{\frac{1}{2}}=-\xi\pm\sqrt{a\xi-\eta^2}. \tag{34}$$

This shows that emission of linear waves is only possible when $a\xi>\eta^2$ and that this emission is concentrated at $\lambda=\lambda_{\frac{1}{2}}$. The group velocity of the linear waves is $v=-4\lambda_{\frac{1}{2}}$ and the maximum of emission occurs when the argument of the cosh is very small, i.e., $a/2-\sqrt{a\xi-\eta^2}\ll 1$ or $a^2/4\approx a\xi-\eta^2$.

The total emitted power $P=\int_{-\infty}^{+\infty}P(\lambda)d\lambda$ is

$$P=\frac{8\pi^3\epsilon^2}{(a\xi)^2}\frac{1}{\sqrt{a\xi-\eta^2}}\left[\frac{F_-^2(\lambda_1)}{\cosh^2[\pi(a-2\sqrt{a\xi-\eta^2})/4\eta]}-\frac{F_-^2(\lambda_2)}{\cosh^2[\pi(a+2\sqrt{a\xi-\eta^2}]/4\eta)}\right]. \tag{35}$$

This formula shows that there is a resonance when $a\xi-\eta^2=0$. To investigate this effect we report numerical experiments done for several initial values of the parameters η and ξ in the plane shown in Fig. 7. The resonance condition $a\xi-\eta^2=0$ is indicated by a solid line. In the first set of experiments we have set initially $\eta=0.5$ and taken three values of the initial velocity ξ: $\xi=0$ (a), which has been presented in Figs. 1 and 2, $\xi=0.25$ (b), and $\xi=0.3$ (c). The temporal evolution of η as a function of time is given in Fig. 8 where case (a) is presented in the top plate, case (b) is presented in the middle plate, and case (c) in the bottom plate. The value η estimated from the numerical experiments is given as a solid line while the dashed line presents the evolution as given by the adiabatic equations (5)–(8). We have good agreement in (a) while in (b) the modulation of the amplitude for the PDE solution tends to increase with time and a big discrepancy is seen. In (c) the modulation for the PDE decreases with time and the soliton will eventually get trapped in a potential well; no good agreement is observed with the solution of the adiabatic equations. These studies are consistent with the result (35), which predicts that no radiation exists for $a\xi-\eta^2<0$ [case (a)] and that there is resonant emission of linear waves for $a\xi-\eta^2=0$ [case (b)] and that this emission exists as soon as $a\xi-\eta^2>0$ [case (c)].

In the second set of experiments we have fixed the initial velocity $\xi=0.25$ and varied the initial amplitude η from $\eta=0.45$ (d), $\eta=0.5$ (b) to $\eta=0.55$ (e); the results are presented from top to bottom in Fig. 9. The PDE solution disagrees with the ordinary differential equation solutions in cases (d) and (b) while there is a reasonable agreement in the case (e) for which $a\xi-\eta^2<0$. This shows that the adiabatic equations (5)–(8) provide an adequate description of the PDE dynamics only when $a\xi-\eta^2<0$. If this condition is not fullfilled the adiabatic description does not agree with the PDE solution and the second-order terms due to the radiation need to be taken into account.

For large velocities $\xi^2\gg a^2,\eta^2$, the total emitted power becomes

$$P=8\pi^3\epsilon^2\xi^4\frac{1}{\sqrt{a\xi-\eta^2}}\left[\frac{1}{\cosh^2[\pi(a-2\sqrt{a\xi-\eta^2})4\eta]}-\frac{1}{\cosh^2[\pi(a+2\sqrt{a\xi-\eta^2})4\eta]}\right]. \tag{36}$$

Notice that when $a\xi\gg\eta^2$, the maximum of emission occurs for $a\approx|4\xi|$, which is the soliton velocity. This is the so-called phase-resonant case found by Scharf and Bishop [6]. We obtain here their estimate directly from inverse scattering theory.

The calculations can also be done for a linear periodic potential $\epsilon \sin axu(x,t)$ and the total emitted power has a very similar expression to the one above

$$P_{\rm linear}=\frac{\pi^3\epsilon^2}{8(a\xi)^2}\frac{1}{\sqrt{a\xi-\eta^2}}\left[\frac{F_-^2(\lambda_1)}{\cosh^2[\pi(a-2\sqrt{a\xi-\eta^2})/4\eta]}-\frac{F_-^2(\lambda_2)}{\cosh^2[\pi(a+2\sqrt{a\xi-\eta^2})/4\eta]}\right], \tag{37}$$

where $F_-=2\eta\Delta(\Delta-a)+\eta^3[-1+3(\Delta_-/\eta)^2]$.

The conclusions that were drawn from the case of modulated dispersion hold in this case too. Therefore the numerical results of Scharf and Bishop [6] can be interpreted in the light of the above resonance condition. When the variations of the potential are large compared to the soliton width, i.e., $a<\eta$ a large velocity ξ is necessary for the soliton to emit radiation. For a small velocity no radiation is emitted so that the medium can be considered as transparent for the soliton. This is in agreement with the results of Kivshar *et al.* [16] which were obtained by taking for V a sum of δ functions and assuming independent scattering. In the case $a>\eta$, the so-called ''dressed'' soliton will emit radiation for very small velocities. Only when $\xi=0$ will there be no emitted radiation.

We will now proceed to show that for large velocities ξ the soliton will experience exponential radiative damping as was evidenced in some early numerical experiments by one of the authors [17]. In the case of a linear potential the number of quanta $N=\int_{-\infty}^{+\infty}|u|^2dx$ is conserved so that we can use the following formula from inverse scattering theory [11]:

$$N=4\eta+\int_{-\infty}^{+\infty}|b(\lambda)|^2d\lambda,$$

where b is the Jost reflection coefficient. Differentiating this expression with respect to time yields

$$0=\frac{dN}{dt}=4\frac{d\eta}{dt}+\int_{-\infty}^{+\infty}P(\lambda)d\lambda.$$

Using expression (37) for the total emitted power and the expression of F_- we obtain the exponential damping of the soliton amplitude

$$\frac{d\eta}{dt}=-\eta\frac{\pi^3\epsilon^2}{32(a\xi)^2}\frac{1}{\sqrt{a\xi-\eta^2}}\left[\frac{G_-^2(\lambda_1)}{\cosh^2[\pi(a-2\sqrt{a\xi-\eta^2})/4\eta]}-\frac{G_-^2(\lambda_2)}{\cosh^2[\pi(a+2\sqrt{a\xi-\eta^2})/4\eta]}\right],$$

where $G_-=2\Delta(\Delta-a)+\eta^3[-1+3(\Delta_-/\eta)^2]$.

The power emitted in the form of linear waves is also important for the study of an array of planar waveguides where the parameter ξ is the tangent of the angle of propagation in the (z,x) plane. We will show in the next section that such a system can be reduced to a perturbed nonlinear Schrödinger equation where the perturbation consists of both a space-dependent dispersion and a linear potential. Therefore for that problem the influence of radiation is important and can lead to a resonant distorsion of the spatial soliton. A detailed knowledge of this damping mechanism could enable one to shift the beam from one guide to another. Figure 10 shows the soliton profile for three different times $t=10$ (top), 20 (middle), and 30 (bottom) for the resonant case. The difference between the solution given by the adiabatic equations as a dashed line and the PDE solution grows as time increases. For $t=30$ the pulse given by the PDE is trapped in a potential well, while the adiabatic equations predict that it keeps moving left. Thus the pulse has been shifted from the position $x=56$ to $x=45$.

V. APPLICATION TO THE PROPAGATION OF SPATIAL SOLITONS IN ARRAYS OF OPTICAL WAVEGUIDES WITH VARIABLE COUPLING

The problem of the propagation of an envelope soliton in a medium with a space-dependent dispersion studied in the previous sections has an important application in the nonlinear optics of waveguides. Let us consider the problem of the propagation of a soliton in an array of tunnel-coupled waveguides with a variable coupling [5,19]. The corresponding system of equations has the form

$$-iu_{nz}=V_{n,n+1}u_{n+1}+V_{n-1,n}u_{n-1}+|u_n|^2u_n. \tag{38}$$

In the continuum limit it is possible to use the Taylor expansion for u_n and $V_{n,m}$:

$$u_{n\pm1}=u(x)\pm hu_x+\frac{h^2}{2}u_{xx}+\cdots,$$

$$V_{n,n\pm1}=V\left(x\pm\frac{h}{2}\right)=V(x)\pm\frac{h}{2}V_x+\frac{h^2}{8}V_{xx}+\cdots.$$

Substituting these expressions into Eq. (38), we obtain for $u(x,z)$ the equation

$$-iu_z=2Vu+h^2(V_xu_x+\tfrac{1}{4}V_{xx}u+Vu_{xx})+|u|^2u+O(h^4). \tag{39}$$

One can define a reduced spatial variable $y=x/h\sqrt{2}$, use the phase shift $v=\rightarrow e^{-4iz}u$, and rescale the variable z by $\frac{1}{2}$ to obtain the reduced equation

$$iv_z+v_{yy}+2|v|^2v=(1-V)v_{yy}-V_yv_y-\tfrac{1}{4}V_{yy}v-4(V-1)v. \tag{40}$$

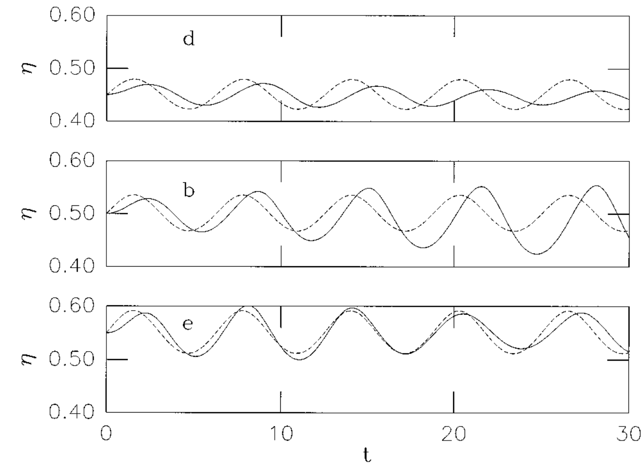

FIG. 9. Time evolution of η for the PDE (solid line) and the adiabatic equations (dashed line) for the positions d, b, and e indicated in Fig. 7 corresponding to initial amplitudes $\eta=0.45$, 0.5, and 0.55, respectively. The values of the parameters are $a=1$, $\epsilon=0.1$ and the initial conditions are $\xi_0=0.25$, $\zeta_0=10\pi$.

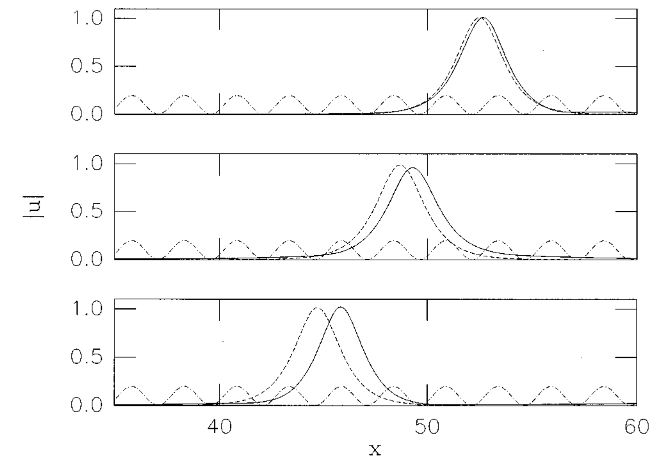

FIG. 10. Profile of $|u|$ as a function of x for different times $t=10$ (top), $t=20$ (middle), $t=30$ (bottom) for the resonant case, the PDE solution is given by the solid line and the solution given by the adiabatic Eqs. (5)–(8) given by the dashed line. The values of the parameters are $a=2.5$, $\epsilon=0.1$ and the initial conditions are $\eta_0=0.5$, $\zeta_0=18\pi$, and $\xi_0=0.1$. $V(x)$ has been drawn as a long-dashed line.

Notice that this system can be derived from the Hamiltonian density

$$\mathcal{H}=V|v_y|^2-|v|^4-[V_{yy}+4(V-1)]|v|^2. \tag{41}$$

In the case of a periodic modulation of the coupling between the waveguides within the array, $V(x)=1+\epsilon\sin(ax)$, Eq. (40) becomes

$$iv_z+v_{yy}+2|v|^2v=-\epsilon(4-\alpha^2/4)\sin(\alpha y)v-\epsilon\alpha\cos(\alpha y)v_y -\epsilon\sin(\alpha y)u_{yy}, \tag{42}$$

where $\alpha=ah\sqrt{2}$.

The adiabatic equations for the soliton parameters can be derived by combining Eqs. (5)–(8) with the ones for a linear potential. The different contributions to the evolution equation of the amplitude cancel out so that

$$\frac{d\eta}{dt}=0. \tag{43}$$

The amplitude of the soliton is not changed. This reflects the fact that the number of quanta $\int_{-\infty}^{+\infty}|u|^2dx$ is conserved by Eq. (39). However, in the case of a chirped beam [3] oscillations of the amplitude and width will appear.

For the velocity (which corresponds to the angle of propagation) we obtain

$$\frac{d\xi}{dt}=\frac{\epsilon\pi\alpha^2\cos(\alpha\zeta)}{2\eta\sinh(\pi\alpha/4\eta)}\left(\frac{\eta^2}{3}+\frac{\alpha^2}{48}+\xi^2-1\right). \tag{44}$$

For the soliton center ζ we obtain the equation

$$\frac{d\zeta}{dt}=-4\xi\left[1+\frac{\epsilon\pi\alpha\sin(\alpha\zeta)}{4\eta\sinh(\pi\alpha/4\eta)}\right]. \tag{45}$$

The fixed points associated to these equations are for ($\epsilon<1$)

$$\xi=0 \quad \text{and} \quad \alpha\zeta=\pm\frac{\pi}{2}+2n\pi,$$

except if $\eta^2/3+\alpha^2/48=1$. It can be seen that the fixed points $\alpha\zeta=\pi/2$ ($-\pi/2$) are stable if $\eta^2/3+\alpha^2/48-1<0$ (>0). Figure 11 shows the phase portraits ($\alpha\zeta,\xi$) obtained for Eqs. (44) and (45) in the case $\eta=1.4$ (top) and $\eta=2$ (bottom) where the orbits are given just as in the case of the pendulum by the level curves of the Hamiltonian

$$H(\zeta,\xi)=16\eta\xi^2+\frac{\epsilon4\pi\alpha\sin\alpha\zeta}{\sinh(\pi\alpha/4\eta)}\left[\frac{\eta^2}{3}+\frac{\alpha^2}{48}+\xi^2-1\right]. \tag{46}$$

One clearly sees that the fixed point ($\alpha\zeta=\pi/2,\xi=0$), which was a center for $\eta=1.4$, becomes a hyperbolic

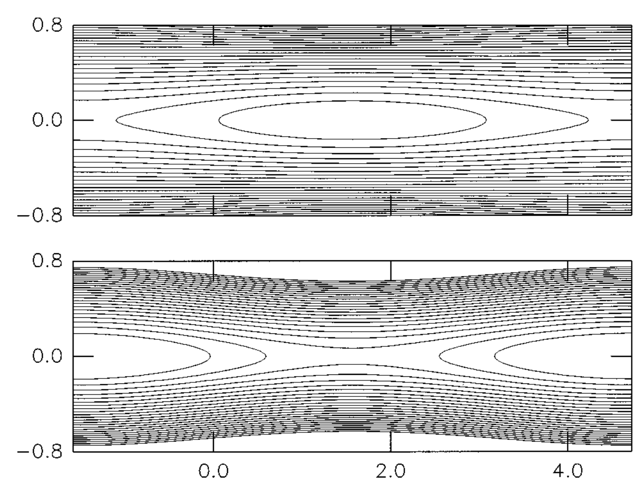

FIG. 11. Phase portraits ($\alpha\zeta,\xi$) for $\eta=1.4$ (top) and $\eta=2$ (bottom) for the adiabatic equations (44)–(45) for an array of waveguides. The values of the other parameters are $\epsilon=0.1$ and $\alpha=1$.

fixed point for $\eta=2$. Notice that when $\eta=\eta^*=\sqrt{3(1-\alpha^2/48)}$ $(\alpha\zeta,\xi=0)$ is a fixed point for any value of ζ so that the solitons of amplitude parameter η^* propagate without being affected by the modulation of the waveguide coupling.

The property of the fixed points described above suggests using such a device as a filter to separate pulses of different amplitudes because the solitons for which $\eta<\eta^*$ get trapped by the fixed points corresponding to $\alpha\zeta=\pi/2$ whereas the pulses for which $\eta>\eta^*$ are attracted to the fixed points corresponding to $\alpha\zeta=-\pi/2$. It would be then possible to use the device as a switch by sending along the channel $\alpha\zeta=\pi/2$ a small pulse so that a pulse that was originally trapped would be shifted to the neighboring channel corresponding to the other type of fixed point.

From the results of Sec. IV it is possible to estimate the condition for resonant decay of the spatial soliton. The incident angle is $\sin\psi_i=2(\xi/\beta)$ where β is the propagation index. Using the condition for resonant emission $a\xi=\eta^2$ we find the value of the incident angle leading to a resonant distortion of the beam in the array of waveguides with a periodically varying coupling

$$\psi_i=a\sin(2\eta^2/\beta a).$$

In the case of a random modulation of the coupling between the waveguides one can estimate the distribution function for the effective random potential relief acting on the soliton in motion. For this we use the technique introduced in [20].

First notice that Eq. (39) can be derived from the Hamiltonian

$$H=\int_{-\infty}^{+\infty}\left\{2V|u|^2+\frac{1}{2}|u|^4+\frac{h^2}{2}\left[\frac{1}{2}V_x(|u|^2)_x-(Vu^*)_xu_x-(Vu)_xu_x^*\right]\right\}dx. \tag{47}$$

Writing $V(x)=V_0+V_1(x)$, one gets

$$H=\int_{-\infty}^{+\infty}\left\{2V|u|^2+\frac{1}{2}|u|^4-\frac{h^2}{2}V_0|u_x|^2+\frac{h^2}{2}\left[\frac{1}{2}V_{1x}(|u|^2)_x-(V_1u^*)_xu_x-(V_1u)_xu_x^*\right]\right\}dx \tag{48}$$

so that for an inhomogeneous array the addition to the energy due to the random modulation is

$$\Delta H=\int_{-\infty}^{+\infty}\left(2V_1-\frac{h^2}{4}V_{1xx}\right)|u|^2dx-\frac{h^2}{2}\int_{-\infty}^{+\infty}[(V_1u^*)_xu_x+(V_1u)_xu_x^*]dx, \tag{49}$$

which reduces by integration by parts to

$$\Delta H=\int_{-\infty}^{+\infty}V_1\left(2|u|^2-\frac{h^2}{2}[|u_x|^2-\mathrm{Re}(u_{xx}u^*)]\right)dx. \tag{50}$$

Let us now find the distribution function for the effective random potential relief acting on the soliton in motion. For this we use the technique introduced in [20] for the sine-Gordon equation. In the case of an inhomogeneous array the addition to the energy due to the random modulation is

$$\Delta H=\int_{-\infty}^{\infty}V_1\left(2|u|^2-\frac{h^2}{2}[|u_x|^2-\mathrm{Re}(u^*u_{xx})]\right)dx. \tag{51}$$

We must find the distribution function $P(\Delta H)$ where

$$P(\Delta H)=\langle\delta(\Delta H-\Delta H(x))\rangle. \tag{52}$$

For the calculation of P we introduce $\Delta H(x)$, defined as

$$\Delta H(x)=\int_{-\infty}^{x}f(x)V_1(x)dx,$$

where

$$f(x)=2|u|^2-\frac{h^2}{2}[|u_x|^2-\mathrm{Re}(u^*u_{xx})], \tag{53}$$

which satisfies the following equation:

$$\frac{\partial\Delta H(x)}{\partial x}=F(x)=f(x)V(x). \tag{54}$$

To obtain the equation for $P(\Delta H,x)$ we will differentiate Eq. (52) by x and get

$$\begin{aligned}\frac{\partial P}{\partial x}&=-\left\langle\frac{\partial\Delta H(x)}{\partial x}\frac{\partial}{\partial\Delta H}\delta(\Delta H-\Delta H(x))\right\rangle\\&=-f(x)\frac{\partial}{\partial\Delta H}\langle V(x)\delta(\Delta H-\Delta H(x))\rangle.\end{aligned} \tag{55}$$

Here we have the correlator $\langle V(x)\delta(\Delta H-\Delta H(x))\rangle\equiv\langle V(x)\delta(\Delta H,x)\rangle$. This correlator can be decoupled by the Furutzu-Novikov formula [14]

$$\langle V(x)\delta(\Delta H,x)\rangle=\int_0^x dx'\langle V(x)V(x')\rangle\left\langle\frac{\partial}{\partial V}\delta(\Delta H,x)\right\rangle. \tag{56}$$

Assuming V is δ correlated we obtain

$$\langle V(x)\,\delta(\Delta H,x)\rangle=\sigma^2\left\langle \frac{\partial}{\partial V}\,\delta(\Delta H,x)\right\rangle. \tag{57}$$

Here we should note that

$$\left\langle \frac{\partial}{\partial V}\,\delta(\Delta H,x)\right\rangle=\left\langle \frac{\partial}{\partial \Delta H}\delta(\Delta H,x)\,\frac{\partial \Delta H}{\partial V}\right\rangle \tag{58}$$

$$=f(x)\,\frac{\partial}{\partial \Delta H}\langle\,\delta(\Delta H,x)\rangle$$

$$=f(x)\,\frac{\partial}{\partial \Delta H}\langle P(\Delta H,x)\rangle. \tag{59}$$

Using Eqs. (57) and (59) we obtain finally the equation for the distribution function $P(\Delta H,x)$:

$$\frac{\partial P(\Delta H,x)}{\partial x}=\sigma_v^2 f^2(x)\frac{\partial^2}{\partial \Delta H^2}P(\Delta H,x). \tag{60}$$

Its solution has the form

$$P(\Delta H,x)=\frac{1}{\sqrt{2\pi\sigma_v^2 G(x)}}\exp\left(-\frac{\Delta H^2}{2\sigma_v^2 G(x)}\right),$$

$$G(x)=\int_{-\infty}^{x} f^2(x')\,dx'. \tag{61}$$

Therefore the additional energy ΔH added to the energy of the soliton because of the presence of the perturbation terms induced by the array has a Gaussian distribution of mean 0 and standard deviation $\epsilon\sqrt{G(\eta,\xi,\infty)}$, where

$$G(\eta,\xi,\infty)=\int_{-\infty}^{\infty} f^2(x')\,dx'=\frac{128}{3}\eta^3\left[1-8\xi^2+16\xi^4 -\frac{16}{5}\eta^2(1-4\xi^2)+\frac{96}{35}\eta^4\right], \tag{62}$$

where the parameter h has been eliminated by the redefinition of x.

The quantity G is a measure of the energy that a soliton can have given the parameters of the potential. For large η and small ξ the pinning potential is

$$U_p=\sqrt{G(\eta,\xi,\infty)}\approx\frac{8\sqrt{2}\,\eta^{7/2}}{\sqrt{3}}. \tag{63}$$

So in a random medium with a small correlation length, the soliton is trapped by an effective potential that has a typical scale of the order of the soliton width.

VI. CONCLUSION

We have performed an analytical and numerical investigation of the dynamics of an envelope soliton in a medium with spatially variable dispersion. For a periodic modulation the PDE solution agrees well with the evolution given by the adiabatic equations for the soliton parameters for small velocities. We also present preliminary results in the random case.

When the velocity is increased the soliton emits linear waves and we calculate the density of emission. We find that there is a resonance when $a\xi-\eta^2=0$ and have investigated the parameter space around that point. Here it is important to recognize the practical usefulness of the inverse scattering theory because this condition cannot be easily obtained from physical considerations. Note also that our study, which is valid both for a linear potential and in the space-dependant dispersion case, shows that linear waves are emitted as soon as $a\xi-\eta^2$ is positive.

We have applied this study to the problem of propagation of spatial solitons in arrays of waveguides with a periodic variation of the coupling and find that in the continuum limit the problem reduces to an NLS equation with a Hamiltonian perturbation. We show that the amplitude of the pulse is conserved and that large (small) amplitude pulses are attracted by the maxima (minima) of the coupling potential, enabling the device to act as a filter or a coupler. Finally we calculate the Gaussian distribution function for the additional energy of the solitons due to the array.

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